Research statement

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My research concerns the theory of orders and the theory of graphs from the reverse mathematics point of view. Reverse mathematics is a vast research program which dates back to the Seventies. Its central question, as formulated in [8], the main monograph in this area, is "Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?". Aiming to answer this question, the purpose of reverse mathematics is to clarify some traditional problems in foundations of mathematics, such as the legitimacy of the use of infinite objects, using tools from mathematical logic. In particular, David Hilbert already noticed that theorems of very different branches of mathematics, such as analysis, algebra and combinatorics, are provable in the formal theory of second order arithmetic. However, proving that a certain theorem follows from some axioms is not sufficient to understand which are the minimal axioms required to prove the theorem; in order to satisfy this minimality requirement it is necessary to prove the axioms themselves from the theorem over a base theory. There are older applications of this process in logic, but reverse mathematics offers a specific formal context and a fixed base theory, which allow to compare theorems of very different areas. Even if the primary focus of reverse mathematics is about the strength of set-existence axioms needed to prove some theorems, in more recent years more attention has been paid to the computational content of these same theorems. More specifically, the relationships among the theorems have also been understood as carrying information about the computational core of some theorems and about the methodological core to which each subsystems of second order arithmetic can be associated (see [4]). Reverse mathematics allows also researchers to create a hierarchy of theorems, which is fruitfully compared with other classifications based more on computability features of the theorems. The benchmarks of the reverse mathematics hierarchy are the following:

- RCA_0: the base theory, where comprehension is limited to Δ^0_1-predicates,
- WKL_0: the extension of RCA_0 obtained adding Weak König's Lemma, i.e. each infinite binary tree has an infinite path,
- ACA_0: allows to form sets defined by arithmetical formulae,
- ATR_0: arithmetical comprehension can be iterated along any well-order,
- Π^1_1-CA_0: allows to form sets defined by Π^1_1-formulae.

Combinatorial problems were analysed since the beginning of reverse mathematics, but there are still plenty of unexplored areas and open questions. Moreover, important results about the strength of combinatorial principles opened new questions and problems in this area, sparking a growing interest for reverse mathematics, computable reducibility and Weihrauch reducibility over the last two decades.

The next sections present the main topics of my research, explaining the results obtained so far and raising questions for future investigation.

1 Reverse mathematics and interval graphs

The topics in this section are linked by the concept of interval graph. Interval graphs form a natural class of graphs, which may be used to represent events on a time line or quantities with a range of error. As their name suggests, interval graphs are graphs whose points can be mapped into intervals of a linear order in such a way that two vertices have an edge in common if and only if the intervals associated to them overlap. More formally, a graph (V, E) is an interval graph if there are a linear order $(L, <_L)$ and a representation $F \subseteq V \times L$ which satisfy the following two conditions (where $F(x)$ has to be read as an abbreviation for the set $\{l \in L \mid (x, l) \in F\}$) for each $x, y \in V$: (a) $F(x)$ is an interval (i.e. $u <_L v <_L w$ and $u, w \in F(x)$ implies $v \in F(x)$ for all $u, v, w \in L$), (b) xEy if and only if $F(x) \cap F(y) \neq \emptyset$.

This is joint work with Alberto Marcone.

1.1 Characterisation of interval graphs

Several characterisations of interval graphs were proposed in the literature (see [2] for a more extensive treatment of interval graphs and other references). Our goal is to understand the relative strength of these characterisations and to compare them with the strength of similar characterisations of interval orders, which were already studied in [6]. Other examples of analyses of structural characterisations of graphs can be found in [5].

In order to give a sample of theorems we proved in this area, we first introduce a couple of definitions. A 1-1 interval graph is an interval graph such that distinct vertices are associated to distinct intervals. More formally, an interval graphs (V, E) is a 1-1 interval graph if there are a linear order $(L, <_L)$ and a representation $F \subseteq V \times L$ which witness that (V, E) is an interval graph and such that if $x \neq y$, for $x, y \in V$, then $F(x) \neq F(y)$. Moreover, an order $(V, <)$ is an interval order if there are a linear order $(L, <_L)$ and a representation $F \subseteq V \times L$ which satisfy the following two conditions (where $F(x)$ has to be read as an abbreviation for the set $\{l \in L \mid (x, l) \in F\}$) for each $x, y \in V$: (a) $F(x)$ is an interval, (b) $x < y$ if and only if $\forall l \in F(x) \forall m \in F(y)(l <_L m)$.

Theorem 1 (RCA_0)**.** WKL_0 *is equivalent to the following: a countable graph* (V, E) *is an interval graph if and only if it is triangulated (i.e. every simple cycle of length four or more has a chord) and has no asteroidal triple (i.e. no set of three distinct vertices with no edge in common such that each pair is connected by a path that avoids the neighbourhood of the third vertex).*

Theorem 2 (RCA_0)**.** WKL_0 *is equivalent to the following: if a countable graph* (V, E) *is an interval graph, then it is a 1-1 interval graph.*

Theorem 3 (RCA_0)**.** WKL_0 *is equivalent to the following: let* $(V, <)$ *be a countable order.* $(V, <)$ *is a 1-1 interval order if and only if* (V, E)*, where* $pEq \Leftrightarrow p \not< q \wedge q \not< p$ *for all* $p, q \in V$*, is a 1-1 interval graph.*

From our results we can conclude that if a graph is an interval graph, namely there are a linear order L and a representation $F \subseteq V \times L$, then it is always possible to find a 1-1 representation, but this cannot be done computably. Hence, in RCA_0 the two concepts, interval graph and 1-1 interval graph, are different. Nonetheless, WKL_0 is enough to prove their equivalence. We analysed other statements similar to the ones mentioned above. Moreover, we analysed the characterisation of a subclass of interval graphs, called indifference graphs, and their relationship with proper interval orders, which are the subclass of interval orders corresponding to indifference graphs.

1.2 Unique orderability of interval graphs

When thinking about comparability graphs, one question that arises naturally concerns the conditions under which a given graph is associated to a unique order up to duality. Notice that interval graphs are complementary graphs of comparability graphs, whose associated orders are interval orders. Moreover, the reversal of Theorem 2 exploits this feature of comparability graphs, namely that there are generally several orders compatible with a given comparability graph.

Some characterisations for unique orderability are known in the literature. However, the one for connected interval graph is stated only for finite graphs, while we are mainly interested in countable graphs. We prove that the same characterisations hold also for infinite interval graphs.

Theorem 4. *A connected infinite interval graph* (V, E) *is uniquely orderable if and only if it does not contain buried subgraphs.*

A buried subgraph B is a set of vertices of V with at least two incomparable elements and such that there is a non empty subset $K \subseteq V \setminus B$ whose points are adjacent to all $b \in B$ and such that each path between $b \in B$ and $v \in V$ contains a $k \in K$.

We also prove that the previous theorem is equivalent to ACA_0. This explains why it is not possible to prove Theorem 4 by compactness, as usually happens in this context, exploiting the finite case.

1.3 Reorientations of pseudo-transitive graphs

The problem of characterising interval graphs led us to analyse the strength of the following statement: each pseudo-transitive oriented graph has a transitive reorientation. An oriented graph $(V, \to)$ is pseudo-transitive if for each vertices a, b, c such that $a \to b \to c$, it holds that $a \to c$ or $c \to a$. Moreover, R is a reorientation of $\to$ if a, b are $\to$-comparable if and only if they are R-comparable. The statement thus asserts that if $\to$ is pseudo-transitive, then it can be reoriented to obtain a transitive R, i.e. an order on V.

One proof of the characterisation theorem for comparability graphs exploits this statement as an essential step towards the conclusion. The idea of the proof is as following: given a non oriented graph

(V, E) with some properties it is possible to define a bipartite graph (W, F) out of (V, E). Thanks to the properties of (W, F), it is not so difficult to show that it is pseudo-transitive orientable and hence transitive orientable. From this information one finally get an order for the original graph (V, E), which witnesses that (V, E) is a comparability graph.

Alain Ghouila-Houri in [3] proved the statement that each pseudo-transitive finite oriented graph has a transitive reorientation, which immediately generalises to infinite graphs by a compactness argument. On the other hand, it was not known if the statement is computably true. We proved the following.

Theorem 5. *There exists an on-line algorithm to transitively reorient (possibly infinite) pseudo-transitive oriented graphs.*

The input of an on-line (incremental) algorithm consists of vertices coming one at a time together with all information about the edges connecting them to previous vertices. When the algorithm sees a new vertex, it must reorient all the edges connecting it to previous vertices while preserving the reorientations already set at previous stages.

It follows that the statement is computably true, and so that the compactness argument used to generalise from the finite to the countable case is not necessary. Moreover, the previous theorem implies that the the multi-valued function that maps a countable pseudo-transitive oriented graph to the set of its transitive reorientations is computable

This statement may be compared with the characterisation of comparability graphs studied in [5]. Jeffrey Hirst proved that WKL_0 is equivalent to the statement that if a graph is such that each cycle of odd length has a triangular chord, then it is a comparability graph. The latter statement essentially asks to find a direction for non-oriented edges of a graph with certain properties. By Hirst's analysis it is not computably true. Theorem 5 and Hirst's theorem calibrate the different strength needed to find an order out of nothing and out of the clues provided by a pseudo-transitive orientation.

In order to prove Theorem 5, we analysed very carefully the properties that a reorientation of a finite subset of V must have in order to be extendible to other vertices of V. This machinery help us to find an algorithm and to prove its correctness.

1.4 Dimension of interval graphs

Lastly, within our research concerning interval graphs, we focused our attention on statements which provide lower and upper bounds on the dimension of families of posets. The dimension of a poset $(P, <_P)$ is the minimum number of linear extensions of $<_P$ whose intersections is equal to $<_P$. Dimension is an important parameter describing a poset, as are height and width. Statements about dimension have not been explored yet in reverse mathematics. We analysed some basic statements, which guarantee for example that in RCA_0 each countable poset has an associated dimension. Moreover, we were able to establish the following.

Theorem 6 (RCA_0)**.** WKL_0 *is equivalent to the following: each countable interval order* $(P, <_P)$ *with height* n *and more than three points has dimension less or equal to* $n + 1$*.*

The previous statement provides an upper bound for dimension of interval orders, which still is not very sharp. For a subclass of interval orders, namely for proper interval orders, the upper bound is much tighter, since it is known that their dimension is never greater than three. This fact is not computably true (see [1]) and is provable in WKL_0.

Question 7. Is the statement that each countable proper interval order has dimension less or equal to three equivalent to WKL_0?

2 Reverse mathematics of Ramsey-theoretic results by Rival and Sands

The second line of research focuses on two statements proved in [7]. All the results mentioned in the next sections are partial outcomes of an ongoing project joint with Alberto Marcone, Paul Shafer and Giovanni Soldà.

2.1 The Rival-Sands theorem for graphs

In 1980 Ivan Rival and Bill Sands [7] proved that each infinite graph G has an infinite subgraph H such that each vertex of G is adjacent to none or to one or to infinitely many vertices of H. It is worth comparing this statement with Ramsey's theorem for pairs. The later statement guarantees that each

infinite graph has a complete or a totally disconnected subgraph. Both statements exhibit a substructure with some nice properties in every infinite graph. Notice that a solution to Ramsey's theorem determines completely the adjacency relation inside the subgraph H, while it lacks information about the adjacency structure of H with respect to the other vertices of G. On the other hand, a solution to the Rival-Sands theorem, while weakening the complete information about H itself, gives some information about the relationship between the interior of H and its exterior, i.e. the rest of G. Indeed, the authors themselves presented the Rival-Sands statement as a variation of Ramsey's theorem which copes with this asymmetry between the information on the inside structure of the solutions and on the outside of them.

In reverse mathematics Ramsey's theorem is the prototype of the so called 'Ramsey's type' principles. A feature of Ramsey's type principles is the fact that each infinite subset of a solution is still a solution. Despite the superficial similarity with Ramsey's theorem, the Rival-Sands theorem is a Ramsey's type principle for some graphs, but not for all of them. For example if the graph is locally finite, each infinite subgraph of a solution is still a solution.

We investigated this statement (restricted to countable graphs) from the viewpoint of reverse mathematics, establishing the following.

Theorem 8 (RCA_0)**.** *The following is equivalent to* ACA_0*: each graph G has an infinite subgraph H such that each vertex of G is adjacent to none or to one or to infinitely many vertices of H.*

Theorem 9 (RCA_0)**.** *If a locally finite graph G is such that there is a computable bound to the degree of the vertices of G, then there is an infinite subgraph H such that each vertex of G is adjacent to none or to one or to infinitely many vertices of H.*

Rival-Sands theorem for graphs revealed to be stronger than Ramsey's theorem for pairs. By the previous theorem, the coding power of the former theorem is at least $0'$, but we suspect that it is higher. This gives rise to the following question.

Question 10. What is the coding power of the Rival-Sands theorem for graphs?

2.2 The Rival-Sands theorem for orders

Rival and Sands wondered if it is possible to be more precise about the properties of the subgraph given as a solution to their theorem. They noticed that there is a graph whose complete and totally disconnected subgraphs are not solutions to the Rival-Sands statement (notice that this observation already implies that Ramsey's theorem for pairs cannot prove the Rival-Sands theorem). Nonetheless, if one restricts to a specific class of graphs, then a more precise answer can be given. This idea pushed Rival and Sands to formulate a variant of the first theorem. In this case an instance of the problem is a poset with finite width (i.e. such that there is a finite bound on the size of each antichain in P), so a particular comparability graph. A solution is an infinite chain C, so a complete subgraph, such that each vertex of P is comparable to none or to infinitely many vertices of C. We are interested in analysing the strength of this statement, RSpo, restricted to countable posets, from the point of view of reverse mathematics. Our starting point was the analysis of the proof in Rival-Sands article, which goes through in Π^1_1-CA_0.

Theorem 11. Π^1_1-CA_0 *proves* RSpo.

A careful reading of the proof reveals the passage where Π^1_1-CA_0 is used. In the proof Rival and Sands made essential use of chains which are maximal among the chains without maximum. If a poset of arbitrary cardinality contains a chain without maximum, then Zorn's lemma assures that there is a chain which is maximal among them. For countable posets we were able to prove the following.

Theorem 12 (RCA_0)**.** *The following is equivalent to* Π^1_1-CA_0*: let $(P, <_P)$ be a poset and $C \subseteq P$ a chain without maximum. Then there is a maximal chain $D \supseteq C$ without maximum.*

Thus the original proof of RSpo makes essential use of Π^1_1-CA_0. The great strength of the principle above is due to the fact that in order to choose which points belongs to D we have to distinguish points with an ω chain above them from points without this property. Here is where Π^1_1-CA_0 comes into play.

Since RSpo is formalisable by a Π^1_2-statement it cannot imply Π^1_1-CA_0, because of established model-theoretic facts. Henry Towsner in [9] showed that some proofs of Π^1_2-statements in Π^1_1-CA_0 actually use a weaker form of maximality, namely one generally only needs that certain objects are maximal with respect to a certain other class of objects, rather than absolutely' maximal. Thus he formulated a hierarchy of principles, called Σ_α-LPP_0, which lie in between Π^1_1-CA_0 and ATR_0.

Question 13. Is the proof of RSpo in Π^1_1-CA_0 actually a proof in Σ_α-LPP_0 for some α?

An answer to the previous question would give a deeper comprehension of the original proof.

Exploiting some relationships between an ascending chain A, whose tails are not solutions, and a chain which witnesses that each tail of A is not a solution, we were able to give a new proof of RSpo, which goes through in ACA_0. Moreover, despite the great strength required by the original proof, we were able to give an entirely different proof of RSpo in ADS getting a sharp result.

Theorem 14. *For each $k \geq 3$,* ADS *is equivalent to* RSpo *for posets of width k.*

ADS is the principle stating that each linear order contains an infinite ascending chain or an infinite descending chain. The structure of this proof is different from the proof in ACA_0, but again it takes inspiration from some combinatorial facts about counterexamples to certain chains being solutions to RSpo. We think that this result is interesting because, as far as we know, RSpo restricted to partial orders of width three is the first theorem of ordinary mathematics proved to be equivalent to ADS. In reverse mathematics ADS received attention as an easy consequence of Ramsey's theorem, which is nonetheless strictly weaker than Ramsey's theorem for pairs, but neither computably true nor provable from WKL_0. ADS shares this behaviour with many other statements, which are quite close, yet non equivalent, to each other. This behaviour contrasts with that of the so called Big Five of reverse mathematics, which are characterised by a sort of robustness and by the equivalence to numerous theorems from different areas of mathematics.

The strength of RSpo for posets of width two is instead strictly weaker than ADS.

Theorem 15. *Over* WKL_0, SADS *is equivalent to* RSpo *for posets of width 2.*

SADS is a weakening of ADS. Since SADS + WKL_0 and ADS are incomparable, it follows that RSpo for posets of width two is strictly weaker than RSpo for posets of width three. The following question arises naturally.

Question 16. Is RSpo restricted to posets of width 2 equivalent to SADS over RCA_0?

Since we are interested in establishing the strength of RSpo, we paid attention to countable posets. Nonetheless, our proofs do not exploit the fact that the poset is countable, hence they also represent proofs of RSpo for posets of arbitrary cardinality, which are entirely different from the original one and with another combinatorial flavour. Moreover, the proof in ADS has the advantage of not employing the axiom of choice, used in the original proof to argue that a maximal chain without maximum exists. On the other hand, the proof in the Rival-Sands article allows to conclude something more about the solution C. In fact, it is actually proven that for each countable poset P of finite width, there is an infinite chain C such that each vertex of P is comparable to none or to co-finitely (not only infinitely) many vertices of C. We call this stronger version of the Rival-Sands theorem sRSpo. Differently from RSpo we get the following result.

Theorem 17. *Over* RCA_0, sRSpo *for posets of width 2 is equivalent to* ADS.

The solutions to RSpo found in the proofs in ACA_0 and in ADS are not always strong solutions. At the moment Π^1_1-CA_0 remains the best upper bound for sRSpo for posets of width greater than two.

Question 18. What is the strength of sRSpo for poset of width $k \geq 3$?

2.3 Variants of CAC

By analysing RSpo, we became interested in the restriction of CAC (i.e. the statement that every poset contains an infinite chain or an infinite antichain) to posets of finite width. In fact, RSpo trivially implies that each poset of finite width has an infinite chain. We named this principle, namely that each infinite poset of finite width has an infinite chain, CC. The initial motivation to study CC was the fact that a lower bound for CC guarantees a lower bound to RSpo as well.

By analogy with CC, we had a look at the principle saying that each poset of finite height has an infinite antichain (we named it CA). We were able to establish the equivalence with $\mathsf{B}\Sigma^0_2$ for both CC and CA.

Theorem 19. *Over* RCA_0, CC *and* CA *are equivalent to* $\mathsf{B}\Sigma^0_2$.

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